ON THE METHOD OF EFFECTIVE NONLINEAR SIGMA MODEL IN PLANE- AND AXIALLY-SYMMETRIC VACUUM SPACETIMES

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In present article effective nonlinear sigma model (NSM) is considered. Einstein equation solution, corresponded to the chiral fields determined by functional parameter method, are presented. Effective NSM of stationary axially-symmetric gravitational field is constructed. Motion equations are solved exactly by functional parameter method. Einstein equations solution are constructed. For particular dependences of functional parameter graphics of solutions are presented. Metric coefficients behaviour is shown to be similar as functional parameter one.

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1 Introduction

In the present article the following block-diagonal metrics are considered

$$ds^2 = g_{ab}dx^a dx^b + g_{\mu\nu}dx^\mu dx^\nu. \tag{1}$$

Metric coefficients are supposed to be depended on x^μ, x^ν. Block $g_{\mu\nu}$ is sure to be transformed to conformal-plane form. Then metrics shall take the form $ds^2 = g_{ab}dx^a dx^b - f\eta_{\mu\nu}dx^\mu dx^\nu$, $\eta_{\mu\nu} = \mathrm{diag}(1,\epsilon)$, $\epsilon = \mathrm{sign}\det g_{ab}$, f depends on x^μ, x^ν only. Signature $(-,-,-,+)$ is accepted. Choosing $\{x^\mu, x^\nu\} = \{z,t\}, \{x^a, x^b\} = \{x,y\}$ leads to plane-symmetric case, $\{x^\mu, x^\nu\} = \{r,z\}, \{x^a, x^b\} = \{t,\phi\}$ corresponds to stationary axially-symmetric spacetimes.

Spacetimes, corresponding to (1) was intensively discussed from different point of view during last 10-20 years.

Inverse scattering method (ISM) technique was applied for Einstein equations solution at first in the work [1]. Cosmological applications was based on solutions mentioned above [2], [3], [4]. Nonlinear transformations put in the base of ISM make impossible, in general, the construction of solution, possessing some definite demanded properties. Analysis of solution obtained is rather difficult due to the same reasons. Some ways to this problem solution were suggested in work [5], [6], where ISM was presented in modified form.

Other avenue of spacetimes (1) investigation is connected with some special solutions construction, such as Gowdy cosmological solutions [8], [9]. The solution obtaining process was aimed at inhomogeneous universe construction.

Method suggested in [10] devoted to symmetry investigation of (1) was applied for exact solution obtaining in works [11], [12], [13], [14]. Some advantage of this method are represented here.

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This way is based on the possibility of construction of effective nonlinear sigma model (NSM)

$$\mathcal{L}_{NSM} = \sqrt{|g|}\{\frac{1}{2}h_{AB}(\varphi)\varphi^A_{,i}\varphi^B_{,k}g^{ik}\}, \tag{2}$$

motion equations of which

$$\frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}\varphi^{;i}_A) - \Gamma_{C,AB}\varphi^B_{,i}\varphi^C_{,k}g^{ik} = 0, \tag{3}$$

appear to be equivalent to vacuum Einstein equations

$$R_{ik} = 0. \tag{4}$$

In work [14] dynamic equations of effective NSM (2), (3) have been solved by the method of functional parameter. The problem of vacuum Einstein equations solution construction hasn't been taken into consideration. Here we will use the method mentioned above for construction and solutions analysis of (4).

Consider two important classes of spacetimes, derived from (1) by appropriated coordinate choosing: plane-symmetric and axially-symmetric.

2 Exact solutions in case of plane-symmetric spacetime

Consider a spacetime

$$ds^2 = Adx^2 + 2Bdxdy + Cdy^2 - D[dz^2 - dt^2], \tag{5}$$

$\sqrt{g} = \alpha D, \alpha^2 \equiv AC - B^2$; functions A,B,C,D depend only on z,t.

Vacuum Einstein equation $R_{ik} = 0$, having been shown [11], are equivalent to effective NSM equations

$$\Box\, e^\psi = 0,$$

$$\Box\, \theta + (\psi_{,z}\theta_{,z} - \psi_{,t}\theta_{,t}) - \frac{1}{2}(\chi_{,z}{}^2 - \chi_{,t}{}^2)\sinh 2\theta = 0,$$

$$\Box\, \chi + (\psi_{,z}\chi_{,z} - \psi_{,t}\chi_{,t})$$

$$+ 2(\theta_{,z}\chi_{,z} - \theta_{,t}\chi_{,t})\coth\theta = 0, \tag{6}$$

$$\Box\,\phi - \frac{1}{2}({\psi_{,z}}^2 - {\psi_{,t}}^2) + \frac{1}{2}({\theta_{,z}}^2 - {\theta_{,t}}^2) + + \frac{1}{2}({\chi_{,z}}^2 - {\chi_{,t}}^2)\sinh^2\theta = 0,$$

$\Box\ \equiv \partial_{zz} - \partial_{tt}$, [11], if metric coefficients are connected with chiral fields as follows

$$\begin{aligned} A &= -e^{\psi}(\cos\chi\sinh\theta + \cosh\theta), \\ B &= e^{\psi}\sin\chi\sinh\theta, \\ C &= e^{\psi}(\cos\chi\sinh\theta - \cosh\theta), \\ D &= e^{\phi}. \end{aligned} \tag{7}$$

Equations (6) have been solved by functional parameter method, suggested in [14]. In this method

$$\begin{aligned} &\psi = \ln\xi, && \theta = \theta(\xi), \\ &\chi = \chi(\xi) && \phi = \phi(\xi), \\ & && \xi = \xi(z,t). \end{aligned} \tag{8}$$

According to the first equation of (6) $\Box\,\xi(z,t) = 0$. This way lead to the following solutions of the (6).

$$\begin{aligned} &\psi = \ln\xi,\ \Box\,\xi(z,t) = 0 \\ &\theta = \mathrm{Arch}\left\{k/2\left((\xi/\xi_0)^a + (\xi/\xi_0)^{-a}\right)\right\} \\ &\chi_\pm = \mathrm{arctg}\chi_0 \\ &\pm\,\mathrm{arctg}\left\{|a/c|\frac{(\xi/\xi_0)^{2a}+1}{(\xi/\xi_0)^{2a}-1}\right\} \\ &\phi = \phi_0 + \phi_1\xi - \frac{a^2-1}{2}\xi,\ k = \sqrt{1+c^2/a^2}. \end{aligned} \tag{9}$$

Here a, c are arbitrary constants.

Other solutions family, suggested in [14], connected with supposition

$$\begin{aligned} &\psi = \mathrm{const}, && \theta = \theta(\xi), \\ &\chi = \chi(\xi), && \phi = \phi(\xi), \\ & && \xi = \xi(z,t). \end{aligned} \tag{10}$$

In this case equations take the form

$$\begin{aligned} &\theta = \pm\mathrm{Arch}\left[\frac{k}{2}\cosh(a(\xi-\xi_0)\right] \\ &\chi = \mathrm{arctg}\chi_0 \pm \mathrm{arctg}\left[\left|\frac{a}{c}\right|\tanh\left\{a(\xi-\xi_0)\right\}\right] \\ &\phi = \phi_0 + \phi_1\xi + \frac{1}{4}a^2\xi^2. \end{aligned} \tag{11}$$

Metrics coefficients, reconstructed by (7) for the case (9) may be represented in such a way

$$\begin{aligned} A &= -\xi\left(\left|\frac{1\mp\chi_0 G(\xi)}{\alpha}\right|\sqrt{\frac{F^2(\xi)-1}{G^2(\xi)+1}} + F(\xi)\right), \\ B &= \xi\left(\left|\frac{\chi_0 \pm G(\xi)}{\alpha}\right|\sqrt{\frac{F^2(\xi)-1}{G^2(\xi)+1}}\right), \\ C &= \xi\left(\left|\frac{1\mp\chi_0 G(\xi)}{\alpha}\right|\sqrt{\frac{F^2(\xi)-1}{G^2(\xi)+1}} - F(\xi)\right), \end{aligned} \tag{12}$$

$$\begin{aligned} F(\xi) &= \frac{k}{2}\left((\xi/\xi_0)^a + (\xi/\xi_0)^{-a}\right), \\ G(\xi) &= \left|\frac{a}{c}\right|\frac{(\xi/\xi_0]^{2a}+1}{(\xi/\xi_0)^{2a}-1},\ \alpha^2 = 1+\chi_0^2. \end{aligned}$$

The form of metric coefficient, associated with the second solutions family, coincidence with (12), but dependencies $F(\xi)$ and $G(\xi)$ are following.

$$F(\xi) = \frac{k}{2}\cosh(a(\xi-\xi_0), \tag{13}$$

$$G(\xi) = \frac{|a|}{|c|}\tanh\left\{a(\xi-\xi_0)\right\} \tag{14}$$

Formally, the only restriction to the form of parameter-function $\xi(z,t)$ is $\xi > 0$. In this case both chiral fields and metric coefficients will be real functions for all $z, t \in \mathbf{R}$.

For instance of solution construction consider the function

$$\begin{aligned} &\xi(z,t) = \exp\left(\cosh^{-2}(z-t)\right) \\ &+\exp\left(\cosh^{-2}(z+t)\right) \end{aligned} \tag{15}$$

for the case (8)-(9). It's known that dependence (15) describes solitons, moving to the opposite direction at the light speed. Graphics of the solution are represented on figure 1. The forms of the metric coefficients at $t = 0$ are shown by solid line and by dotted line at $t = 3$.

Solution for $g_{11} = A(z,t)$, $g_{12} = B(z,t)$, $g_{22} = C(z,t)$, $g_{33} = D(z,t)$ conservative the soliton-like character. Impulse corresponding to $D(z,t)$ influence the considerable amplitude decries since $t = 0$ to $t = 1$, surpassing expected one according to (15). During the further evolution form of metric coefficients experience no changes.

Possibility of $\xi(z,t)$ choice allows to construct solutions of the Einstein vacuum equation possessing the determined properties. This is a advantage of functional parameter method over Belinskii-Zakharov method [1], which set no direct connection between seed solution and solution obtained.

3 Effective nonlinear sigma model of axially-symmetric spacetime

In order to apply this method to axially-symmetric spacetimes analysis it is necessary to construct effective NSM in this case.

Consider spacetime in the following form

$$\begin{aligned} ds^2 &= e^{2\nu(z,r)}dt^2 - e^{2\mu(z,r)}(dz^2+dr^2) \\ &- e^{2\rho(z,r)}(d\phi - \omega(z,r)dt)^2. \end{aligned} \tag{16}$$

Connection between effective chiral fields and metric coefficients should be defined by the way allowing

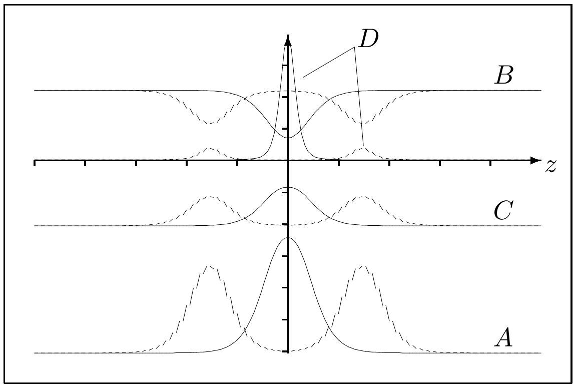

Figure 1: Gravitation fields at $t = 0$ and $t = 3$ $a = 2, c = 1, \xi_0 = 10, \chi_0 = 10$.

i)representation of the Legrangian of gravitation field (16) $\mathcal{L}_G = \sqrt{-g}g^{ik}\left(\Gamma^l_{km}\Gamma^m_{il} + \Gamma^l_{ik}\Gamma^m_{lm}\right)$ as effective NSM Legrangian (2);

ii)transformation of Einstein equations to motion equations of the corresponding effective NSM.

These conditions are satisfied both in the following case.

Relation between chiral fields and metrics coefficients is determined in such a way

$$\begin{array}{l} \mathrm{e}^{2\nu} = 2\mathrm{e}^{\psi}\cosh\theta, \omega = -\dfrac{\sinh\chi}{\cosh\chi - \coth\theta} \\ \mathrm{e}^{2\mu} = \mathrm{e}^{\phi}, \mathrm{e}^{2\rho} = -\mathrm{e}^{\psi}(\cosh\chi\sinh\theta - \cosh\theta). \end{array} \tag{17}$$

Fields are defined on space $dS^2 = dz^2 + dr^2$. Intrinsic NSM space is chosen in form

$$h_{IK} = e^{\psi}\begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \sinh^2\theta & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{18}$$

Dynamical equations in this case will be

$$\begin{array}{l} \triangle\,\mathrm{e}^{\psi} = 0, \\ \triangle\,\theta + (\psi_3\theta_{,z} - \psi_{,r}\theta_{,r}) + \dfrac{1}{2}({\chi_{,z}}^2 - {\chi_{,r}}^2)\sinh 2\theta = 0, \\ \triangle\,\chi + (\psi_{,z}\chi_{,z} - \psi_{,r}\chi_{,r}) \\ \qquad + 2(\theta_{,z}\chi_{,z} - \theta_{,r}\chi_{,r})\coth\theta = 0, \\ \triangle\,\phi - \dfrac{1}{2}({\psi_{,z}}^2 - {\psi_{,r}}^2) + \dfrac{1}{2}({\theta_{,z}}^2 - {\theta_{,r}}^2) \\ \qquad - \dfrac{1}{2}({\chi_{,z}}^2 - {\chi_{,r}}^2)\sinh^2\theta = 0, \end{array} \tag{19}$$

$\triangle \equiv \partial_{zz} + \partial_{rr}$.

4 Exact solutions

Solution will be south by functional parameter method. According to its ideology consider

$$\begin{array}{ll} \psi = \ln\xi(z,r) & \theta = \theta(\xi), \\ \chi = \chi(\xi) & \phi = \phi(\xi). \end{array} \tag{20}$$

The ordinary differential equation set will be achieved

$$\begin{array}{l} \ddot{\theta} + \dfrac{1}{\xi}\dot{\theta} + \dfrac{1}{2}\dot{\chi}^2\sinh 2\theta = 0, \\ \ddot{\chi} + \dfrac{1}{\xi}\dot{\chi} + 2\dot{\chi}\dot{\theta}\coth\theta = 0, \\ \ddot{\phi} - \dfrac{1}{2\xi^2} + \dfrac{1}{2}\dot{\theta}^2 + \dfrac{1}{4}\dot{\chi}^2\sinh^2\theta = 0 \end{array} \tag{21}$$

Here and henceforth $\dot{\varphi}_i \equiv \dfrac{\partial\varphi_i}{\partial\xi}$ Solutions of (21) depending on ξ will posses form

$$\begin{array}{rcl} \theta & = & \mathrm{Arch}\left\{k\sin\left|\ln\left(\xi_0/\xi\right)^a\right|\right\} \\ \chi & = & \mathrm{Arth}\chi_0 \pm \dfrac{a}{\sqrt{c^2-a^2}} \\ & & \times\,\mathrm{arctg}\left\{\pm\dfrac{c}{\sqrt{c^2-a^2}}\tan\left|\ln\left(\xi_0/\xi\right)^a\right|\right\} \\ \phi & = & \phi_0 + \phi_1\xi + \dfrac{a^2-1}{2}(\ln\xi + 1) - \xi\displaystyle\int^{\xi}\dfrac{\chi(\xi')}{\xi'^2}d\xi', \\ k & = & \sqrt{1 + c^2/a^2},\ a^2 < c^2,\ c > 0,\ a < 0. \end{array} \tag{22}$$

Metric coefficients may be obtained by (17).

$$\begin{array}{l} \mathrm{e}^{2\nu} = 2\xi F(\xi) \\ \mathrm{e}^{2\rho} = \xi F(\xi) - \xi\left|\dfrac{1 \pm \chi_0\tanh G(\xi)}{\alpha}\right| \\ \times\sqrt{\dfrac{F^2(\xi)-1}{1-\tanh^2 G(\xi)}}, \\ \omega = -(\chi_0 \pm \tanh G(\xi)) \\ \times\left(\left|1 \pm \chi_0\tanh G(\xi)\right| - |\alpha|F(\xi)\sqrt{\dfrac{1-\tanh^2 G(\xi)}{F^2(\xi)-1}}\right)^{-1}. \end{array} \tag{23}$$

$$\begin{array}{rcl} F(\xi) & = & k\sin\left|\ln\left(\xi_0/\xi\right)^a\right|, \\ G(\xi) & = & \dfrac{a}{c^2-a^2}\mathrm{arctg}\left[\pm\dfrac{c}{\sqrt{c^2-a^2}}\tan\left|\ln\left(\xi_0/\xi\right)^a\right|\right], \\ \alpha^2 & = & (1-\chi_0)^2. \end{array}$$

There are some conditions for function-parameter, necessary to existence chiral fields and metric coefficients.

$$\begin{array}{l} \dfrac{\arcsin k^{-1} + 2\pi n}{|a|} \le \left|\ln\dfrac{\xi(z,r)}{\xi_0}\right| < \dfrac{\pi(2n+\frac{1}{2})}{|a|}, \\ \cosh\chi(\xi) < \coth\theta(\xi). \end{array} \tag{24}$$

The first inequality of (24) can't be solved analytically. So it's impossible to determine the conditions of restrictions for $\xi(z,r)$ in general case with arbitrary constants. As soon as constants-parameters are defined inequalities may be solved. Solutions of (24) determine $\xi(z,r)$ value set.

Next class of solution may be constructed by using the similar suggestion as in (10).

$$\begin{aligned} \psi &= \text{const}, \qquad \theta = \theta(\xi), \\ \chi &= \chi(\xi) \qquad \phi = \phi(\xi),\ \triangle\,\xi = 0. \end{aligned} \tag{25}$$

Set of ODEs in this case will be written as follows

$$\begin{aligned} &\ddot{\theta} + \frac{1}{2}\dot{\chi}^2 \sinh 2\theta = 0, \\ &\ddot{\chi} + 2\dot{\chi}\dot{\theta}\coth\theta = 0, \\ &\ddot{\phi} + \frac{1}{2}\dot{\theta}^2 + \frac{1}{4}\dot{\chi}^2 \sinh^2\theta = 0. \end{aligned} \tag{26}$$

It's solution

$$\begin{aligned} \theta &= \text{Arch}\left\{k \sin|a(\xi - \xi_0)|\right\} \\ \chi &= \text{Arth}\chi_0 \pm \frac{a}{\sqrt{c^2 - a^2}} \\ &\quad \times \text{arctg}\left\{\pm\frac{c}{\sqrt{c^2 - a^2}} \tan|a(\xi - xi_0)|\right\} \qquad (27) \\ \phi &= \phi_0 + \phi_1\xi + \frac{1}{4}a^2\xi^2 - \int^{\xi} \chi(\xi')d\xi', \\ k &= \sqrt{1 + c^2/a^2},\ a^2 < c^2,\ c > 0,\ a < 0. \end{aligned}$$

The form of metrics coefficients will be the same as (23), but $F(\xi)$ and $G(\xi)$ will be determined in other way

$$\begin{aligned} F(\xi) &= k \sin|a(\xi - \xi_0)|, \\ G(\xi) &= \frac{a}{c^2 - a^2}\text{arctg}\left\{\pm\frac{c}{\sqrt{c^2 - a^2}} \tan|a(\xi - \xi_0)|\right\}, \\ \alpha^2 &= (1 - \chi_0)^2 \end{aligned}$$

The solution existence conditions will look like discussed above.

$$\begin{aligned} &\frac{\arcsin k^{-1} + 2\pi n}{|a|} \le |\xi(z, r) - \xi_0| < \frac{\pi(2n + \frac{1}{2})}{|a|}, \\ &\cosh\chi(\xi) < \coth\theta(\xi). \end{aligned} \tag{28}$$

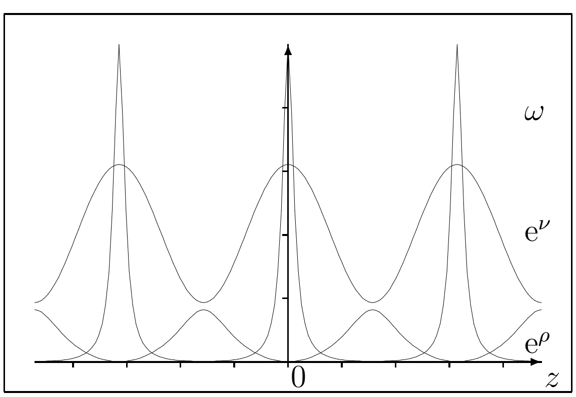

Figure 2: Gravitation fields distribution on axis ($r = 0$)

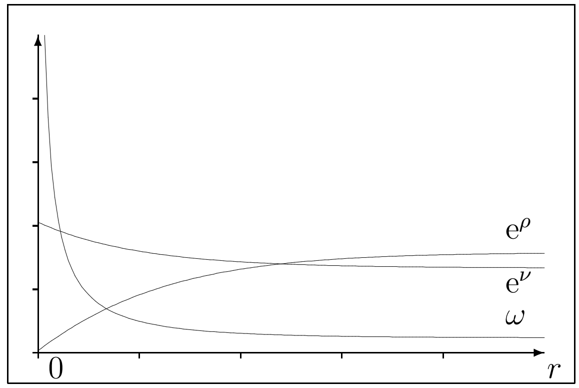

Figure 3: Gravitation fields distribution in plane $z = 0$

Consider the method application in the case (20-23). Let choice function-parameter in the form $\xi(z, r) = b(e^{-r}\cos z + d)$. Coefficients b and d are determined with respect to of conditions (24).

Graphics of solutions, achieved by choosing $a = -1, c = 1000, \xi_0 = 1, \chi_0 = 0.1$ are presented on figures 2 and 3.

In the regions located along the axis z gravitational fields distribution possesses periodical character. In the plane, crossing axis on $z = 0$ metric coefficients rapidly decries to some constant value. Such a behaviour is determined by the form of $\xi(z, r)$. Evidently, desire fields distribution may be achieved by matching form of function-parameter.

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